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LETTER TO THE EDITOR

On the relationship between a 2 × 2 matrix and second-order scalar spectral problems for integrable equations

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Abstract

A simple proof is given that a 2 × 2 matrix scheme for an inverse scattering transform method for integrable equations can be converted into the standard form of the second-order scalar spectral problem associated with the same equations. Simple formulae relating these two kinds of representation of integrable equations are established.

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It is well known that many integrable partial differential equations can be expressed as compatibility conditions of two linear systems represented by 2 × 2 matrices. First introduced by Zakharov and Shabat [1] for integration of the nonlinear Schrödinger (NLS) equation, this approach was generalized by Ablowitz *et al* [2] and by Wadati *et al* [3] to many other integrable equations. We shall write the 2 × 2 systems in the form

$$\begin{aligned} &\Psi_x = \mathbb{U}\Psi \qquad \Psi_t = \mathbb{V}\Psi \\ &\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \mathbb{U} = \begin{pmatrix} F & G \\ H & -F \end{pmatrix} \qquad \mathbb{V} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \end{aligned} \tag{1}$$

where the matrix elements depend on the field variables of the equations under consideration, and so the spectral parameter λ. The condition $\Psi_{xt} = \Psi_{tx}$ yields at once the evolution equations in the general form

$$\begin{aligned} &F_t - A_x + CG - BH = 0 \\ &G_t - B_x + 2(BF - AG) = 0 \\ &H_t - C_x + 2(AH - CF) = 0 \end{aligned} \tag{2}$$

which may be specified by the proper choice of the matrix elements. Some general properties of these equations can easily be formulated in terms of the so-called 'squared basis functions' (see, e.g., [4]). If (ψ_1^+, ψ_2^+) and (ψ_1^-, ψ_2^-) denote two basis solutions of the linear systems (1), then we define the squared basis functions by the formulae

$$f = -\frac{\mathrm{i}}{2}\left(\psi_1^+\psi_2^- + \psi_1^-\psi_2^+\right) \qquad g = \psi_1^+\psi_1^- \qquad h = -\psi_2^+\psi_2^-. \tag{3}$$

As follows from (1), they satisfy the equations

$$\begin{aligned} f_x &= -\mathrm{i}Hg + \mathrm{i}Gh & f_t &= -\mathrm{i}Cg + \mathrm{i}Bh \\ g_x &= 2\mathrm{i}Gf + 2Fg & g_t &= 2\mathrm{i}Bf + 2Ag \\ h_x &= -2\mathrm{i}Hf - 2Fh & h_t &= -2\mathrm{i}Cf - 2Ah \end{aligned} \tag{4}$$

which have the integral of motion

$$f^2 - gh = P(\lambda) \tag{5}$$

where the integration constant can only depend on the spectral parameter λ. (In particular, periodic solutions of the evolution equations are distinguished by the condition that $P(\lambda)$ must be a polynomial in λ.) It is easy to prove also with the use of equations (2) and (4) the following identities:

$$\left(\frac{G}{g}\right)_t = \left(\frac{B}{g}\right)_x \qquad \left(\frac{H}{h}\right)_t = \left(\frac{C}{h}\right)_x \tag{6}$$

which can be considered as generating functions of the conservation laws of the evolution equations under consideration. Another form [3] of the generating function of conservation laws can be formulated in terms of the ratio of two basis functions:

$$\Gamma = \psi_2/\psi_1. \tag{7}$$

As follows from (1), this function satisfies the equations

$$\Gamma_x = H - 2F\Gamma - G\Gamma^2 \tag{8}$$

$$(F + G\Gamma)_t = (A + B\Gamma)_x \tag{9}$$

where equation (8) yields the recursion relation for coefficients of the series expansion of Γ in inverse powers of λ and hence equation (9) gives the generating function for the conservation laws.

On the other hand, there are many equations (e.g. the Korteweg–de Vries and Kaup–Boussinesq equations) whose integrability follows most directly from the compatibility condition of two scalar equations:

$$\psi_{xx} = \mathcal{A}\psi \qquad \psi_t = -\tfrac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x \tag{10}$$

where again the coefficients $\mathcal{A}$ and $\mathcal{B}$ depend on the field variables and the spectral parameter λ, with the result that the condition $(\psi_{xx})_t = (\psi_t)_{xx}$ yields

$$\mathcal{A}_t - 2\mathcal{B}_x\mathcal{A} - \mathcal{B}\mathcal{A}_x + \tfrac{1}{2}\mathcal{B}_{xxx} = 0. \tag{11}$$

Now the squared basis function

$$\tilde{g} = \psi^+\psi^- \tag{12}$$

is built from two basis solutions $\psi^\pm$ of the second-order differential equation. It satisfies the equations

$$\tilde{g}_{xxx} - 2\mathcal{A}_x\tilde{g} - 4\mathcal{A}\tilde{g}_x = 0 \qquad \tilde{g}_t = \mathcal{B}\tilde{g}_x - \mathcal{B}_x\tilde{g} \tag{13}$$

which have the first integral

$$\tfrac{1}{2}\tilde{g}\tilde{g}_{xx} - \tfrac{1}{4}\tilde{g}_x^2 - \mathcal{A}\tilde{g}^2 = \tilde{P}(\lambda) \tag{14}$$

where again the integration constant in the right-hand side can only depend on the spectral parameter λ, and the second equation (13) gives at once the generating function for the conservation laws:

$$\left(\frac{1}{\tilde{g}}\right)_t = \left(\frac{\mathcal{B}}{\tilde{g}}\right)_x . \tag{15}$$

The ratio

$$\tilde{\Gamma} = \psi_x/\psi \tag{16}$$

satisfies the equations

$$\tilde{\Gamma}_x = \mathcal{A} - \tilde{\Gamma}^2 \tag{17}$$

$$\tilde{\Gamma}_t = \left(\mathcal{B}\tilde{\Gamma} - \tfrac{1}{2}\mathcal{B}_x\right)_x \tag{18}$$

where (17) yields the recursion relation for coefficients of the series expansion in powers of λ^{-1} and (18) gives the generating function for the conservation laws.

The similarity of these two sets of formulae for the cases of a matrix, on one hand, and a scalar, on the other, is quite evident and it is natural to ask whether a connection exists between these two representations of integrable equations. Of course, it is known from general theory of ordinary differential equations (ODE) that any system of N first-order ODE can be converted into one scalar ODE of the Nth order, and *vice versa*. However, transformation of these scalar equations into standard form (10) requires redefinition of an unknown function and turns out to be quite cumbersome. We have not succeeded in finding any publications with simple and general proofs of equivalence of these two forms of spectral problem (1) and (10) and the corresponding compatibility conditions. For a particular case of the NLS equation, such a relationship was studied by Alber [5] and the connection of the inverse scattering problem with an energy-dependent potential with the Zakharov–Shabat inverse scattering problem was discussed by Jaulent and Miodek [6]. The aim of this letter is to provide simple formulae which directly connect the matrix problem (1) with the second-order scalar problem (10). These formulae permit one to translate all results from one form to another and, in particular, they have direct application to quasiclassical analysis of asymptotic solutions of wave equations described by the 2×2 matrix scheme [7].

A clue to finding the desired relationship between equations (1) and (10) is provided by a comparison of the expressions for the solutions of these equations in terms of the squared basis functions. It is easy to check that the function

$$\psi^{\pm} = \sqrt{\tilde{g}}\,\exp\left(\pm\mathrm{i}\sqrt{\tilde{P}(\lambda)}\int^x \frac{\mathrm{d}x}{\tilde{g}}\right) \tag{19}$$

satisfies the first equation (10) provided that $\tilde{P}(\lambda)$ satisfies equation (14). An analogous expression has been found by one of us in the recent letter [8] for solutions of equations (1):

$$\psi_1^{\pm} = \sqrt{g}\,\exp\left(\pm\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{G}{g}\,\mathrm{d}x\right) \tag{20}$$

where $P(\lambda)$ satisfies equation (5), and a similar formula can be written for $\psi_2^{\pm}$ in terms of h and H:

$$\psi_2^{\pm} = \sqrt{-h}\,\exp\left(\pm\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{H}{h}\,\mathrm{d}x\right). \tag{21}$$

It is easy to see that we can make equations (19) and (20) coincide with each other by putting

$$g = G\tilde{g} \qquad \psi_1^{\pm} = \sqrt{G}\,\psi^{\pm} \tag{22}$$

provided that the constants $P(\lambda)$ and $\tilde{P}(\lambda)$ in equations (5) and (14), respectively, are also the same. To make them equal to each other, we express f and h in terms of $\tilde{g}$ with the use of equations (4) and (22):

$$\begin{aligned} f &= -\frac{\mathrm{i}}{2}\left[(G_x/G - 2F)\tilde{g} + \tilde{g}_x\right] \\ h &= \frac{1}{G}\left\{\frac{1}{2}\left[(2F - G_x/G)_x\tilde{g} - (2F - G_x/G)\tilde{g}_x + \tilde{g}_{xx}\right] + GH\tilde{g}\right\} \end{aligned} \tag{23}$$

and substitute (22) and (23) into equation (5) to obtain

$$\tfrac{1}{2}\tilde{g}\tilde{g}_{xx} - \tfrac{1}{4}\tilde{g}_x^2 - \left[(F - G_x/2G)^2 + GH + (F - G_x/2G)_x\right]\tilde{g}^2 = P(\lambda). \tag{24}$$

Comparison with equation (14) shows that $P(\lambda)$ and $\tilde{P}(\lambda)$ become equal if we take

$$\mathcal{A} = (F - G_x/2G)^2 + GH + (F - G_x/2G)_x. \tag{25}$$

In a similar way, comparison of the first equation (6) with (15) yields

$$\mathcal{B} = B/G. \tag{26}$$

To complete the conversion of equations (1) into the form (10), we have to prove that the compatibility conditions (2) and (11) are also equivalent to each other. To this end, we express A and C in terms of the other matrix elements by means of the first two equations (2) and substitute these expressions into the third equation (2). Then straightforward calculation shows that the resulting equation multiplied by G is equal to equation (11) with $\mathcal{A}$ and $\mathcal{B}$ given by equations (25) and (26), respectively. Thus, the matrix system (1) is transformed into the scalar form (10) where $\mathcal{A}$ and $\mathcal{B}$ are expressed in terms of matrix elements of (1) by the formulae (25) and (26), respectively.

It is easy to show that the second equation (22) leads to the transformation formula

$$\tilde{\Gamma} = F - G_x/2G + G\Gamma \tag{27}$$

which together with (2), (25) and (26) transforms equations (17) and (18) into equations (8) and (9), respectively.

In a similar way, equation (19) can be reduced to (21) by

$$h = H\tilde{g} \qquad \psi_2^{\pm} = \mathrm{i}\sqrt{H}\,\psi^{\pm} \tag{28}$$

and then instead of equations (25), (26) we obtain

$$\mathcal{A} = (F + H_x/2H)^2 + GH - (F + H_x/2H)_x \tag{29}$$

$$\mathcal{B} = C/H \tag{30}$$

what restores the symmetry between G and H.

Let us illustrate the proven relationship with several examples.

As the first example, we take the NLS equation

$$\mathrm{i}u_t + u_{xx} \pm 2|u|^2u = 0 \tag{31}$$

for which the relevant matrix elements in (1) are given by

$$F = -\mathrm{i}\lambda \qquad G = \mathrm{i}u \qquad H = \pm\mathrm{i}u^* \qquad B = 2\mathrm{i}u\lambda - u_x \tag{32}$$

where the two signs correspond to the focusing and defocusing nonlinearities, respectively. Then according to equations (25) and (26), the NLS equation (31) can be presented as a compatibility condition for equations (10), where

$$\begin{aligned} \mathcal{A} &= -(\lambda - \mathrm{i}u_x/2u)^2 \mp |u|^2 - (u_x/2u)_x \\ \mathcal{B} &= 2\lambda + \mathrm{i}u_x/u. \end{aligned} \tag{33}$$

These formulae coincide exactly with those found by Alber [5].

As a less familiar example, we take the derivative NLS (DNLS) equation

$$\mathrm{i}u_t + u_{xx} - 2\mathrm{i}(|u|^2 u)_x = 0 \tag{34}$$

for which [9]

$$\begin{aligned} &F = -2\mathrm{i}\lambda^2 \qquad G = 2\lambda u \qquad H = 2\lambda u^* \\ &B = 8\lambda^3 u + \lambda(2\mathrm{i}u_x + 4|u|^2 u) \end{aligned} \tag{35}$$

and hence

$$\begin{aligned} \mathcal{A} &= -(2\lambda^2 - \mathrm{i}u_x/2u)^2 + 4\lambda^2|u|^2 - (u_x/2u)_x \\ \mathcal{B} &= 4\lambda^2 + 2|u|^2 + \mathrm{i}u_x/u. \end{aligned} \tag{36}$$

The NLS and DNLS equations written in 'complexified' form are related to each other by a gauge transformation found in [10]. Let us show that this transformation is consistent with corresponding scalar forms of spectral problems. Now we write the complexified NLS and DNLS equations in the notation of [10], so that the NLS equations

$$\begin{aligned} &\mathrm{i}q_{1t} + q_{1xx} - 2r_1 q_1^2 = 0 \\ &\mathrm{i}r_{1t} - r_{1xx} + 2r_1^2 q_1 = 0 \end{aligned} \tag{37}$$

correspond to the matrix form (1) with

$$F_1 = -\mathrm{i}\lambda^2 \qquad G_1 = q_1 \qquad H_1 = r_1 \qquad B_1 = \mathrm{i}q_{1x} + 2\lambda^2 q_1 \tag{38}$$

and the DNLS equations

$$\begin{aligned} &\mathrm{i}q_{2t} + q_{2xx} - \mathrm{i}(r_2 q_2^2)_x = 0 \\ &\mathrm{i}r_{2t} - r_{2xx} - \mathrm{i}(r_2^2 q_2)_x = 0 \end{aligned} \tag{39}$$

correspond to (1) with

$$\begin{aligned} &F_2 = -\mathrm{i}\lambda^2 \qquad G_2 = q_2 \qquad H_2 = r_2 \\ &B_2 = 2\lambda^3 q_2 + \lambda(\mathrm{i}q_{2x} + r_2 q_2^2). \end{aligned} \tag{40}$$

These two spectral problems are connected by the gauge transformation [10] leading to the following connection between the field variables:

$$\begin{aligned} q_1 &= (q_2/2)\exp\left(-\mathrm{i}\int^x r_2 q_2\,\mathrm{d}x\right) \\ r_1 &= (-\mathrm{i}r_{2x} + r_2^2 q_2/2)\exp\left(\mathrm{i}\int^x r_2 q_2\,\mathrm{d}x\right). \end{aligned} \tag{41}$$

Equations (25) and (26) give the corresponding forms of scalar spectral problems with

$$\begin{aligned} \mathcal{A}_1 &= -(\lambda^2 - \mathrm{i}q_{1x}/2q_1)^2 + q_1 r_1 - (q_{1x}/2q_1)_x \\ \mathcal{B}_1 &= 2\lambda^2 + \mathrm{i}q_{1x}/q_1 \end{aligned} \tag{42}$$

for the NLS equations (37), and with

$$\begin{aligned} \mathcal{A}_2 &= -(\lambda^2 - \mathrm{i}q_{2x}/2q_2)^2 + \lambda^2 q_2 r_2 - (q_{2x}/2q_2)_x \\ \mathcal{B}_2 &= 2\lambda^2 + r_2 q_2 + \mathrm{i}q_{2x}/q_2 \end{aligned} \tag{43}$$

for the DNLS equations (39), respectively. Now simple calculation shows that formulae (42) transform after substitution of (41) into formulae (43) as one should expect for consistency of matrix and scalar representations.

As the last example, we consider the uniaxial ferromagnet equation

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + J(\boldsymbol{S} \cdot \boldsymbol{n})(\boldsymbol{S} \times \boldsymbol{n}) \tag{44}$$

where $\boldsymbol{S} = (S_1, S_2, S_3)$ is the classical magnetization vector of unit length, $|\boldsymbol{S}| = 1$, and $\boldsymbol{n}$ is a unit vector along the axis of easy magnetization. In this case we have [11]

$$\begin{aligned} &F = -\frac{\mathrm{i}}{2}\lambda S_3 \qquad G = -\frac{\mathrm{i}}{2}\sqrt{\lambda^2 + J}S_- \qquad H = -\frac{\mathrm{i}}{2}\sqrt{\lambda^2 + J}S_+ \\ &B = \frac{\mathrm{i}}{2}\lambda\sqrt{\lambda^2 + J}S_- + \frac{1}{2}\sqrt{\lambda^2 + J}[(S_3)_x S_- - S_3(S_-)_x] \end{aligned} \tag{45}$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$, and hence

$$\begin{aligned} &\mathcal{A} = -\tfrac{1}{4}(\lambda S_3 - 2\mathrm{i}(S_-)_x/S_-)^2 - \tfrac{1}{4}(\lambda^2 + J)S_+S_- - \frac{\mathrm{i}}{2}(\lambda S_3 - 2\mathrm{i}(S_-)_x/S_-)_x \\ &\mathcal{B} = -\lambda + \mathrm{i}S_-(S_3/S_-)_x. \end{aligned} \tag{46}$$

In the limit $J = 0$ we obtain the classical Heisenberg model which is also connected with the NLS equation by gauge transformation [10, 12]. It is easy to derive analogous formulae for many other equations. Generally speaking, the resulting second-order equation corresponds to a complex 'potential'. An example of inverse scattering transform theory with a complex potential is given in [13].

In conclusion, we note that scalar representation is more convenient for studying the weak-dispersion (quasiclassical) limit and corresponding asymptotic solutions of wave equations [7], whereas the symmetric matrix form is better suited for obtaining periodic solutions and Whitham modulational equations (see, e.g., [4]). The connection established here between the two representations permits one to use the advantages of both methods.

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